Physical basis of the Shields curve

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To clarify the fundamental mechanisms that govern the onset of sediment transport, we performed molecular dynamics simulations of granular beds sheared by a model fluid flow. In addition to changing the fluid-driven stress applied to the bed, we varied the grain-grain interactions by including dissipation, friction, and irregular grain shape to determine their effect on the transition from static to moving sediment beds. When we include large collisional dissipation, as would be the case for grain-grain interactions in a viscous fluid, we find a single boundary in parameter space between systems with and without grain motion that is nearly independent of the fluid velocity profile or the grain properties. Our results are consistent with the Shields curve, and therefore help to explain both its grain-scale origins and its robustness over substantial variation in grain properties and channel geometry.

I. INTRODUCTION

A fluid that flows over a granular bed exerts a shear stress on the grains and, if the flow is sufficiently strong, will entrain grains into the flow. This process is responsible for shaping much of the natural world. Understanding and controlling the erosion of sediments by flowing water are significant for a range of ecological and agricultural problems [1–4]. Thus, the nature of the failure of a static granular bed due to a fluid flow has been the subject of extensive research dating back many decades (for example, see recent reviews by Dey [5] and Buffington and Montgomery [6]), but is still not fully understood. This problem involves nontrivial coupling between several physical processes that are each independently difficult to characterize. Predicting the yield stress of granular materials is challenging, even for very simple cases like frictionless disks [7]. In the natural world, the geological processes that produce the granular materials in question yield grains with varying size, shape, roughness, and other material properties [8]. The mechanics of the flow that imparts stress to the bed are also nontrivial, given both the wide range of channel geometries in natural streams and rivers [9, 10] and possibly turbulent conditions. Additionally, the fluid inside the bed is also moving, as the bed can be viewed as a porous material, and is governed by Darcy flow [11], although with a complicated boundary condition linking it to the turbulent flow at the bed surface.

Despite the apparent complexity of this problem, there is evidence that certain aspects, such as the boundary in parameter space between mobile and static beds, can be described relatively simply. In particular, a collection of data dating back over a century suggests that the onset of grain motion can be captured by only two nondimensional parameters [5, 6, 12–24]: the Shields number $\Theta$, which compares the horizontal shear stress exerted by the fluid at the bed surface to the downward gravitational stress acting on these grains, and a particle-scale Reynolds number $Re_p$, which compares the inertia of an entrained grain to the viscous stresses in the fluid. (We note that the Stokes number $St$ more explicitly compares inertia by including the fluid and grain mass densities, but, since these mass densities are not significantly varied for sediment transport, $St$ and $Re_p$ capture the same information.) The minimum Shields number $\Theta = \Theta_c$ required for grain motion as a function of $Re_p$ is often referred to as the Shields curve. Figure 1 shows data taken from Dey [5], who compiled data from a range of other sources [12–24]. These data were collected over a wide range of different flows, spanning the range from laminar to fully turbulent, and for many channel geometries and grain properties. Although the data are scattered, they cluster around a curve with plateaus of $\Theta_c$ at $Re_p \ll 1$ and $\Theta_c^h$ at $Re_p \gg 1$, where $\Theta_c^h > \Theta_c^l$, and a transitional region near $Re_p \approx 1$. But understanding why the Shields curve takes this particular form and why that form is so robust against variation of other parameters, such as the Reynolds number of the flow, the properties of the fluid boundary layer near the bed, the geometry of the channel, or the physical characteristics of the grains, remain open questions.

To better understand the shape and nature of the boundary in parameter space separating systems with and without grain motion, previous studies have used a range of approaches, which can be broadly grouped into three categories. The first involves laboratory-scale experiments, often using tabletop flumes with model sediments (such as glass beads) driven by water. These experiments typically show good agreement with field re-
results; many of the data points shown in Fig. 1(a) were taken from such experiments (e.g., [20]). Experiments have the advantage of yielding more precise grain-scale data than can be acquired in field observations, revealing interesting effects near $\Theta_c$ such as long transient times [26], intermittency of grain motion [27], subsurface creep [28], grain rearrangements leading to bed strengthening [29], and the role of the impulse (rather than simply the stress) delivered by turbulent fluctuations [30]. However, the parameter space of any individual experiment is inherently constrained, and the level of detail for grain-scale or fluid measurements, though improved over large-scale or field experiments, is still limited. Researchers have therefore also conducted numerical simulations of the fluid and grain motion [31–36]. The treatment of each phase varies from highly resolved, where direct numerical simulation (DNS) for the fluid is coupled with the discrete element method (DEM) for the grains, to a coarser approach, where the fluid flow is modeled (either fully, as with a model fluid field, or partially, as with large-eddy simulations) and the granular bed is treated as a continuum by averaging over the microscopic grain properties. Coarser models can be predictive when tuned to experimental or field results, but they explicitly exclude grain-scale physics. The DNS-DEM approach provides arbitrarily fine resolution, and they typically reproduce a value for $\Theta_c(Re_p)$ within the scatter of the Shields curve. Such computations, however, are extremely expensive and thus prohibit full exploration of the parameter space related to fluid flow and grain properties. Additionally, the increased physical realism, and therefore the complexity of the simulation, often obscures the specific mechanisms that control the behavior near $\Theta_c(Re_p)$. Finally, a third approach uses hydraulic or empirical scaling formulas to describe the rate of sediment transport or the shape of the Shields curve [5, 37–39]. These approaches are based in grain-scale physics and can be highly predictive for situations to which they are tuned. However, they often require multiple fit parameters as inputs.

The approach we take here is to use simple numerical simulations to study the basic physical mechanisms that govern the onset of sediment transport. Although previous DEM-based approaches have typically included as much physical realism as possible, our approach is to simplify the problem to include only the most fundamental elements. The validity of this approach can then be checked by comparing the results of the model to experimental data. We identify three fundamental elements for modeling the behavior of fluid-sheared granular beds near the onset of grain motion. First, the granular beds must be unconfined at the top surface (unlike many studies of purely granular flows, which are typically conducted at constant pressure or constant volume). Second, the beds are driven by the flow of a fluid, meaning that the driving in the system is not constant force (as in granular flow down an inclined plane) or constant velocity (as studied commonly for confined shear flows). Instead, the degree of the forcing depends on the state of the grains, since they are coupled, and is relatively large above the bed, very small in the bed, and smoothly connected in between. And finally, if grains are moving, the force on an individual grain depends on the fluid drag, which in turn depends on the relative velocity between the grain and the fluid (i.e., the grain can equilibrate to the fluid flow). The detailed form of the drag force (for example, linear or quadratic in the relative velocity) will likely affect the dynamics for moderate $Re_p$, but the behavior should be independent of the form of the drag force law in the limits of low $Re_p$, where grains equilibrate immediately to the fluid velocity, and high $Re_p$, where grains never equilibrate to the fluid flow and instead feel constant horizontal acceleration set by $\Theta$.

In recent work [25], we studied computationally the simplest model that retains these properties, where elastic, frictionless disks under gravity are driven by a model fluid shear flow that is small in densely packed regions (i.e., in the bed) and large in sparsely packed regions.

FIG. 1. (a) A collection of experimental and field data from [5] showing the variation the minimum Shields number for grain motion $\Theta_c$ with $Re_p$. This data indicates plateaus of $\Theta^c$ at $Re_p \ll 1$ and $\Theta^h$ at $Re_p \gg 1$, which are marked with solid lines. (b) Results from two-dimensional simulations [25] using frictionless, elastic disks and a model fluid force. Blue circles and green squares represent mobile systems that did and did not stop, respectively. Red crosses show the value of $\Theta$ where a static system becomes mobilized. The dashed line shows $\Theta^h$, the minimum value of $\Theta$ at which static systems can become mobilized at high $Re_p$; mobilization always occurs at $\Theta = \Theta^h$ in the large system limit. The solid line connecting $\Theta^c$ and $\Theta^h$ is $\Theta_c(Re_p)$, which marks the minimum $\Theta$ that can maintain grain motion indefinitely at varying $Re_p$. 

\begin{align*}
\Theta^c & \quad \Theta^h \\
\text{at } Re_p & \ll 1 \quad \text{at } Re_p \gg 1
\end{align*}
(i.e., above the bed). The grains are coupled to the fluid through a linear (Stokes-like) drag force. This model excludes many of the complexities that are certainly present in natural systems. Nevertheless, we found curves $\Theta_c(Re_p)$, denoting when mobile grains would stop, and $\Theta_0(Re_p)$, denoting when static beds would become mobilized, as shown in Fig. 1(b). At small $Re_p$, $\Theta_c > \Theta_0$, so there is a single boundary between static and mobile beds in the long-time limit. At large $Re_p$, $\Theta_c < \Theta_0$, which yields hysteretic behavior. In the results we present here, we find that adding friction and significant grain-grain dissipation yields a single boundary separating mobile and static systems (i.e., the difference between $\Theta_c(Re_p)$ and $\Theta_0(Re_p)$ at large $Re_p$ disappears). The shape of this boundary is consistent with the Shields curve shown in Fig. 1(a), and it is largely insensitive to the details of the fluid flow or grain interactions, i.e., changing these properties keeps the boundary curve within the scatter in Fig. 1(a). Additionally, our previous study shows how the results change when tangential contact forces from other particles, a gravitational force, and a Stokes-drag-like force from a fluid that moves horizontally, so that

$$m_i \frac{d\vec{\nu}_i}{dt} = \sum_j F_{ij}^c - m_i g' \vec{y} + B_i [v_0 f(\vec{r}) \hat{x} - \vec{v}_i].$$

The total torque on each particle is only due to tangential contact forces, so that

$$I_i \frac{d\omega_i}{dt} = \sum_j \vec{s}_{ij} \times \vec{F}_{ij}^c.$$  

Here, the sum over $j$ only includes particles contacting particle $i$, $\vec{s}_{ij}$ is the vector connecting the center of particle $i$ to the point of contact between particles $i$ and $j$, $m_i \propto D_i$ is the mass of the particle, $I_i$ is the moment of inertia of the particle, $D_i$ is the diameter of the particle, $v_i$ is the velocity of the particle, $m_i g'$ is the buoyancy-corrected particle weight, $B_i \propto D_i$ sets the drag on particle $i$, $v_0$ is a characteristic fluid velocity at the surface of a stationary bed, and $f(\vec{r})$ is the fluid velocity at $\vec{r}$. For the frictionless, elastic case [25], $F_{ij}^c = F_{ij}^c$, where $F_{ij}^c = K \left(1 - \frac{r_{ij}}{D_i}ight) \theta \left(1 - \frac{r_{ij}}{D_j}ight) \hat{r}_{ij}$ is the pairwise repulsive contact force on disk $i$ from disk $j$, where $K$ is the particle stiffness, $r_{ij}$ is the separation between the centers of the particles, $D_{ij} = (D_i + D_j)/2$, $\hat{r}_{ij}$ is the unit vector connecting their centers, and $\theta$ is the Heaviside step function. In this study, we modify the interparticle contact force $F_{ij}^c$ to include dissipative grain-grain interactions and tangential forces. The dissipative force is given by $F_{ij}^d = \gamma_o \frac{m_i m_j}{m_i + m_j} (\vec{v}_i - \vec{v}_j) \cdot \hat{r}_{ij}$, where the dissipation rate $\gamma_o = \frac{2 \log \pi e_o}{\tau_c}$, $\tau_c = \frac{\pi \sqrt{m}}{K}$ is the grain-grain collision time, and $e_o$ is the coefficient of normal restitution [40]. This form for the normal dissipation is often used to model energy losses that arise from contact mechanics, such as viscoelasticity, internal heating, or internal vibrational modes of grains. Here, it is likely that the fluid in the interparticle gap dominates the energy loss during a collision, and the effective $e_o$ depends on the relative impact velocity of the particles and the viscosity of the fluid [41, 42]. We hold $e_o$ fixed for each individual simulation, independent of the relative grain velocity at contact or local fluid behavior.

Tangential forces in granular beds arise via two mechanisms: non-spherical grain shape and microscopic friction.

II. METHODS

We study systems of $N/2$ large and $N/2$ small grains with diameter ratio 1.4. Our domain of width $W$ has periodic boundaries in the horizontal direction. We use no upper confining boundary and a rigid lower boundary with infinite friction so that the horizontal velocities of all particles touching it are fixed to zero. We integrate Newton’s equations of motion for each particle, including rotational and translation degrees of freedom. The total force on each particle is given by the vector sum of contact forces from other particles, a gravitational force, and a Stokes-drag-like force from a fluid that moves horizontally, so that

$$m_i \frac{d\vec{\nu}_i}{dt} = \sum_j F_{ij}^c - m_i g' \vec{y} + B_i [v_0 f(\vec{r}) \hat{x} - \vec{v}_i].$$

The total torque on each particle is only due to tangential contact forces, so that

$$I_i \frac{d\omega_i}{dt} = \sum_j \vec{s}_{ij} \times \vec{F}_{ij}^c.$$  

Here, the sum over $j$ only includes particles contacting particle $i$, $\vec{s}_{ij}$ is the vector connecting the center of particle $i$ to the point of contact between particles $i$ and $j$, $m_i \propto D_i$ is the mass of the particle, $I_i$ is the moment of inertia of the particle, $D_i$ is the diameter of the particle, $\vec{v}_i$ is the velocity of the particle, $m_i g'$ is the buoyancy-corrected particle weight, $B_i \propto D_i$ sets the drag on particle $i$, $v_0$ is a characteristic fluid velocity at the surface of a stationary bed, and $f(\vec{r})$ is the fluid velocity at $\vec{r}$. For the frictionless, elastic case [25], $F_{ij}^c = F_{ij}^c$, where $F_{ij}^c = K \left(1 - \frac{r_{ij}}{D_i}ight) \theta \left(1 - \frac{r_{ij}}{D_j}ight) \hat{r}_{ij}$ is the pairwise repulsive contact force on disk $i$ from disk $j$, where $K$ is the particle stiffness, $r_{ij}$ is the separation between the centers of the particles, $D_{ij} = (D_i + D_j)/2$, $\hat{r}_{ij}$ is the unit vector connecting their centers, and $\theta$ is the Heaviside step function. In this study, we modify the interparticle contact force $F_{ij}^c$ to include dissipative grain-grain interactions and tangential forces. The dissipative force is given by $F_{ij}^d = \gamma_o \frac{m_i m_j}{m_i + m_j} (\vec{v}_i - \vec{v}_j) \cdot \hat{r}_{ij}$, where the dissipation rate $\gamma_o = \frac{2 \log \pi e_o}{\tau_c}$, $\tau_c = \frac{\pi \sqrt{m}}{K}$ is the grain-grain collision time, and $e_o$ is the coefficient of normal restitution [40]. This form for the normal dissipation is often used to model energy losses that arise from contact mechanics, such as viscoelasticity, internal heating, or internal vibrational modes of grains. Here, it is likely that the fluid in the interparticle gap dominates the energy loss during a collision, and the effective $e_o$ depends on the relative impact velocity of the particles and the viscosity of the fluid [41, 42]. We hold $e_o$ fixed for each individual simulation, independent of the relative grain velocity at contact or local fluid behavior.

Tangential forces in granular beds arise via two mechanisms: non-spherical grain shape and microscopic friction.
FIG. 2. (a) A depiction of the particle-asperity model [43, 44]. Particle clusters are composed of \( n \) frictionless disk-shaped asperities, each with diameter \( d \), with their centers regularly spaced on a circle of radius \( a \). Arrows show the direction of contact forces, and solid lines connect the contact point to the center of the cluster. The angle \( \psi \) between the arrows and solid lines sets the ratio of tangential \( F_t \) and normal \( F_n \) components of contact forces, and thus the maximum ratio \( \max(F_t/F_n) = \mu_{\text{eff}} \) is set purely by geometry. (b,c) Packings constructed from (b) particle clusters and (c) disk-shaped particles with Cundall-Strack friction [45], both containing 25 particles, using an athermal packing generation protocol [46]. Particle clusters shown here have \( n = 5 \) and \( a/d = 0.6 \), which gives \( \mu_{\text{eff}} = 0.6 \) for two particles of the same size. The center particle in panel (a) is contacting the particle to its left with \( F_t/F_n = \mu_{\text{eff}} \). The disk-shaped particles with Cundall-Strack friction have \( \mu = 0.6 \) to match the particle clusters. Panels (d) and (e) show histograms of the ratio \( F_t/F_n \) for all contact forces in (b) and (c), respectively.

We approximate these two mechanisms using a particle-asperity model [43, 44] and the Cundall-Strack model for friction [45], respectively. In the particle-asperity model, shown in Fig. 2(a)-(b) and Fig. 3(b), we use clusters of \( n \) disks of a fixed size \( d \). The centers of the disks lie on a circle of radius \( a \), spaced at angular intervals of \( 2\pi/n \). The non-overlapping area \( A_i \) of each cluster is calculated using Monte Carlo methods, and the effective diameter is set to \( D_i = \sqrt{4A_i/\pi} \). We assume that small disks on each particle cluster interact via purely repulsive linear spring forces. These forces do not generally act through the center of mass of the cluster, and therefore generate torques. (This means that the sum over \( j \) in Eq. (2) now includes multiple contacts between clusters \( i \) and \( j \).) In this way, macroscopic geometrical friction is introduced via the asperities, as is the case in natural systems, where sand or gravel particles are almost never spherical.

In the second approach, shown in Figs. 2(c) and 3(c), particles are represented by disks that interact via Cundall-Strack friction [45], which approximates microscopic friction through the use of linear tangential springs at interparticle contacts with tangential force \( F_t^{ij} = -K_t u_t^{ij} \), where \( K_t = K/3 \) and \( u_t^{ij} \) is the relative displacement of the point of contact between particles \( i \) and \( j \). At each contact, we enforce the Coulomb sliding condition, \( F_t^{ij} \leq \mu F_n^{ij} \), where \( \mu \) is the static friction coefficient. When \( F_t^{ij} \) exceeds \( \mu F_n^{ij} \), we set \( u_t^{ij} = \mu F_n^{ij}/K_t \), and the particles slide relative to each other.

For the case of the particle-asperity model, the value of \( F_t^{ij}/F_n^{ij} \) is determined by local geometry at the points of contact, with \( 0 \leq F_t^{ij}/F_n^{ij} \leq \mu_{\text{eff}} \), where \( \mu_{\text{eff}} \) corresponds to the case of an asperity from one cluster contacting two asperities from a different cluster. For the Cundall-Strack model, the value of \( F_t^{ij}/F_n^{ij} \) depends on the history of the contact, i.e., the accumulated tangential displacement \( u_t^{ij} \). Figure 2 shows a comparison of packings generated with \( \mu = \mu_{\text{eff}} = 0.6 \). The distributions of \( F_t^{ij}/F_n^{ij} \) are similar for the two models; both have a maximum near 0.6 and a broad distribution below. (Note that \( \mu_{\text{eff}} \) for contacts between two clusters of the same size is 0.6,
whereas $\mu_{\text{eff}}$ can be larger for a small cluster contacting a large cluster.

We choose the fluid velocity profile $f(r)$ to vary smoothly from a large value above the bed to a small value inside the bed, according to the local packing fraction of grains $\phi_i$: $f(\phi_i) = e^{-b(\phi_i - 0.5)}$, where $b$ controls the ratio of the magnitude of the fluid flow above and inside the bed. $\phi_i$ is calculated in a small circular region with diameter $D_i + 2D_i$ around each particle. Since $f = 1$ for $\phi_i = 0.5$, a typical value at the bed surface, $v_0$ is roughly equal to the fluid velocity at the free granular surface.

We set the nondimensional thickness $K/m'g > 3 \times 10^3$ to be sufficiently large that our results become independent of $K$. In addition to $K/m'g$, $\mu$, and $e_n$, which determine grain-grain interactions, there are two remaining nondimensional numbers that can be defined from Eq. (1):

$$\Theta = \frac{Bv_0}{mg'},$$

$$\Gamma = \frac{B/m}{\sqrt{g'/D}}.$$  \hspace{1cm} (3)

The Shields number $\Theta$ gives the dimensionless shear force at the top of a static bed. $\Gamma$ is the ratio of the gravitational settling time $\tau_s = \sqrt{D/g'}$ to the viscous time scale $m/B$. Combining these numbers yields a particle Reynolds number or the Stokes number (St):

$$\frac{\Theta}{\Gamma} = \frac{m_{\text{eff}}}{mg'} \propto \frac{\rho_f}{\rho_g} \text{Re}_p \propto \text{St}.$$  \hspace{1cm} (4)

We use this ratio to compare the inertia of grains entrained in the flow to the strength of the viscous damping from the fluid.

To characterize the onset and cessation of bed motion in our system, we employ two protocols. To study the mobile-to-static transition defined by $\Theta_c$, we distribute particles randomly throughout the domain and set a constant value of $\Theta$ for a total time of roughly $10^6$ grain-grain collision times. We then observe if and when our system ever stops, which we define as when the maximum acceleration $a_{\text{max}} < a_{\text{thresh}}$ and maximum velocity $v_{\text{max}} < v_{\text{thresh}}$, where $a_{\text{thresh}}$ is roughly one order of magnitude smaller than $g'$ and roughly three orders of magnitude smaller than typical values for a moving bed and $v_{\text{thresh}}$ is roughly three orders of magnitude smaller than the fluid velocity at the surface. To understand the dynamics of the static-to-mobile transition, we begin with a static bed and slowly increase $\Theta$ in small increments until we observe $a_{\text{max}} > a_{\text{thresh}}$ or $v_{\text{max}} > v_{\text{thresh}}$. We then keep $\Theta$ constant until $a_{\text{max}} < a_{\text{thresh}}$ and $v_{\text{max}} < v_{\text{thresh}}$ or until the end of our simulation.

### III. RESULTS

#### A. Frictionless, elastic grains

In previous work using frictionless, elastic disks, the protocols described in Section II revealed two different boundaries [25]. We defined $\Theta_c(\text{Re}_p)$ as the minimum value of $\Theta$ for each $\text{Re}_p$ at which systems with moving grains will eventually stop, and we observed that the time for grains to stop diverges as $\Theta$ approached $\Theta_c$ from below. We found $\Theta_c$ to be independent of system size. We defined $\Theta_0(\text{Re}_p)$ as the minimum value of $\Theta$ required for the system to transition from a static state to one with substantial grain motion. We observed two different kinds of behavior at large and small $\text{Re}_p$. At large $\text{Re}_p$, $\Theta_c < \Theta_0$, so static systems that began to move above $\Theta_0$ never stopped. At small $\text{Re}_p$, $\Theta_c > \Theta_0$, so bed flows above $\Theta_0$ were temporary, with large variations in size. The particular value $\Theta = \Theta_f$ where a certain system became permanently mobile was distributed according to Weibullian weakest-link statistics [47, 48], meaning that the global failure of any particular configuration was in fact caused by local failure, and increasing system size caused the distribution of $\Theta_f$ to approach $\Theta_c$ for low $\text{Re}_p$ and $\Theta_0$ for high $\text{Re}_p$.

We observed that $\Theta_c$ and $\Theta_0$ plateau in the limits of high $(\Theta_{c,0}^h)$ and low $\text{Re}_p$ $(\Theta_{c,0}^l)$. Here we will focus on characterizing the plateau values $\Theta_c^l$, $\Theta_c^h$, and $\Theta_0^h$, as shown in Fig. 1(b). We neglect $\Theta_0(\text{Re}_p)$ at low $\text{Re}_p$ when $\Theta_0 < \Theta_c$. Finding $\Theta_0(\text{Re}_p)$ in this region requires choosing a minimum grain motion that counts as a temporarily mobile bed, which is somewhat arbitrary.

In Clark et al. [25], we showed that our results were qualitatively insensitive changes in the fluid flow profile. In our model, $b$ controls the shape of the fluid profile by setting the relative strength of flow rates above the bed and inside the bed. We studied $b = 2$, 4, and 6, and we observed that $\Theta_c(\text{Re}_p)$ showed some dependence on $b$, although it’s basic shape was unchanged. In the present study, we focus exclusively on the case of $b = 5$, which corresponds to a fluid flow that is roughly 7.4 times larger above the bed than at the surface, and a fluid flow that
FIG. 3. Panels (a-c) show snapshots of simulations using (a) frictionless disks; (b) particles clusters from Fig. 2 with \( \mu_{\text{eff}} = 0.6 \); and (c) disks with Cundall-Strack friction \( [45] \), with \( \mu = 0.6 \). All three simulations shown here are at \( \Theta = 0.25 \) and \( \text{Re}_p \approx 1.6 \), with \( c_n = 0.8 \). The vertical axis gives the height \( y/D \) above the lower boundary, where \( D \) is the particle diameter, and the horizontal axis gives the horizontal velocity \( v_x/v_0 \) of grains and the fluid, where \( v_0 \) is the characteristic fluid velocity at the bed surface. Solid and dashed lines show the time-averaged grain velocity \( v_g \) and fluid velocity \( v_f \), respectively.

TABLE II. The flow velocity ratios and plateau values \( \Theta^l_c \), \( \Theta^h_c \), and \( \Theta^h_0 \) for frictionless elastic disks as a function of the fluid decay constant \( b \). \( v_a \), \( v_0 \), and \( v_b \) are the fluid velocities above the bed, at the surface of the bed, and in the interior of the bed, respectively.

<table>
<thead>
<tr>
<th>( b )</th>
<th>( v_a/v_0 )</th>
<th>( v_0/v_b )</th>
<th>( \Theta^l_c )</th>
<th>( \Theta^h_c )</th>
<th>( \Theta^h_0 )</th>
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<tr>
<td>2</td>
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<td>2.0</td>
<td>0.5</td>
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<td>7.7</td>
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</table>

is roughly 5.5 times smaller inside the bed than at the surface. The values of the fluid velocity ratios are shown in Table II as a function of \( b \).

We perform simulations spanning roughly six orders of magnitude in \( \text{Re}_p \) by setting \( \Gamma = 5, 2, 1, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01, \) and 0.005. For the largest and smallest values of \( \Gamma \), our results are relatively insensitive to \( \Gamma \). We define \( \Theta^l_c \) and \( \Theta^h_c \) as the minimum \( \Theta \) at which grains will never stop in the respective low and high \( \text{Re}_p \) limits, and \( \Theta^h_0 \) as the minimum \( \Theta \) at which we observe permanent grain motion in the high \( \text{Re}_p \) limit. These plateau values are given as a function of \( b \) in Table II, and plotted in Fig. 4. We note that \( b = 5 \) means that the flow inside the bed is sufficiently small that \( \Theta^l_c \) is independent of \( b \) (that is, stability at low \( \text{Re}_p \) is dominated by the top layer of grains). This regime corresponds to a fluid force in the bulk of the bed that is at least 5 times weaker than at the surface, a condition that is certainly met in experiments and in nature. \( \Theta^h_0 \) is the value of \( \Theta \) at which highly inertial static grains that begin to move will lead to permanent bed motion, and it is nearly independent of \( b \), since, as we will show in Section IV, the behavior is dominated by the dynamics of grains on the bed surface under a constant acceleration. \( \Theta^h_c \) is the value of \( \Theta \) at
which highly inertial mobilized grains will stop, and it depends strongly on $b$, since mobilized grains above the bed will obtain more energy for larger fluid flow above the bed. This observation on its own suggests that a more accurate fluid flow model above the bed may be necessary to properly capture the behavior at large $\text{Re}_p$. However, we will show below that once we include grain-grain dissipation, this effect is greatly diminished, and $\Theta_n \rightarrow \Theta_n^0$.

The plots of the boundaries marking the static-to-mobile transition, and vice versa, we show throughout the remainder of this manuscript were obtained by running roughly 1000 simulations with $N = 200$ and initial values of $0.02 < \Theta < 1.4$ and $10^{-2} < \text{Re}_p < 10^4$. $\Theta^c_n$ and $\Theta^h_n$ are determined by finding the minimum value of $\Theta$ where a mobile system will stop at low and high $\text{Re}_p$, respectively. $\Theta^c_n$ is determined by slowly increasing $\Theta$ for the systems that stop at high $\text{Re}_p$ (roughly 200 simulations for any given set of grain properties) and finding the minimum value of $\Theta$ at which a static system will become mobile. Since our measurement of $\Theta^h_n$ was obtained in a finite system, we may overestimate the true value by a few percent.

### B. Grain-grain dissipation

As we stated in the Section I, one of our primary goals in this study is to understand how the results of our model change when the energy loss at moderate to high $\text{Re}_p$ is not dominated by the bottom wall but instead by more realistic dissipative grain-grain interactions in a viscous fluid. Our main result is that the boundaries $\Theta^c_n(\text{Re}_p)$ and $\Theta^h_n(\text{Re}_p)$ remain qualitatively unchanged with grain-grain dissipation, but that $\Theta^h_n$ increases and approaches $\Theta^h_n$ for small $e_n$ [41, 42]. Since $\Theta^h_n$ is independent of the fluid velocity profile (Fig. 4), $\Theta^h_n \rightarrow \Theta^h_n$ may explain why the data in Fig. 1(a) is so robust over so many different systems. Additionally, we find that systems with and without dissipation are characterized by exponential decay of a mobile layer into a static bed, as shown in Fig. 6. In this sense, adding collisional dissipation does not qualitatively change the motion of the grains, since the decay of the mobilized layer of all systems we studied has roughly the same functional form (exponential decay) and can thus be rescaled onto a master curve.

Figure 5(a) shows results for frictionless disks with $e_n = 0.8$. We again find a curve $\Theta^c_n(\text{Re}_p)$ that determines when mobile beds will stop, with plateaus $\Theta^c_n$ and $\Theta^h_n$ at low and high $\text{Re}_p$, respectively. We plot the values of these plateaus as a function of $e_n$ in Fig. 5(b). As we decrease $e_n$, we observe that $\Theta^h_n$ (when static beds become mobile at high $\text{Re}_p$) and $\Theta^h_n$ (the boundary between static and mobile beds at low $\text{Re}_p$) are unchanged, meaning that the values associated with these plateaus do not depend on the elasticity of grain-grain collisions. However, as $e_n$ approaches zero, $\Theta^h_n$ increases and approaches $\Theta^h_n$, so that the size of the hysteresis $\Theta^h_n - \Theta^h_n$ decreases with restitution coefficient $e_n$. As $e_n$ approaches zero, mobilized grains undergo perfectly inelastic collisions with the bed, meaning that they do not rebound into the larger fluid flow and they become much less efficient in mobilizing additional grains. Yang and Hunt [41] and Joseph et al. [42] showed that collisions between spheres that are made from materials that would yield large $e_n$ in dry collisions (e.g., steel) can have very small $e_n$ when the collisions are mediated by a viscous fluid.

Figure 4 shows that the plateau $\Theta^h_n$ is virtually independent of the fluid velocity profile, but that $\Theta^h_n$ depends strongly on the profile above the bed. Since $\Theta^h_n$ is insensitive to the shape of the velocity profile above the bed, and since $\Theta^h_n$ approaches $\Theta^h_n$ as $e_n$ decreases from $e_n = 1$ to more realistic values, our findings can explain the collapse of the data in Fig. 1(a) over so many different kinds of systems.

We observe that systems with and without grain-grain dissipation exhibit a mobilized layer that decays exponentially into the static bed, where the decay length depends how the energy is dissipated. A moderate amount of collisional dissipation is sufficient to make grain-grain dissipation...
interactions dominate the energy loss instead of damping from the fluid, causing the decay from the mobile layer to static bed to occur over roughly one grain diameter into the bed, independent of \( \text{Re}_p \). Figures 6(a) and (b) show the average packing density profile \( \phi \), and Fig. 6(c) shows the average horizontal velocity profile \( v_x \) of mobilized grains. These quantities decay exponentially as \( \phi_0 - \phi \propto \exp(-y/l_\phi D) \) and \( v_x \propto \exp(-y/l_{v_x} D) \), where \( l_\phi \) and \( l_{v_x} \) represent decay lengths into the bed. For frictionless elastic disks [25], \( l_\phi \) and \( l_{v_x} \) depend strongly on \( \text{Re}_p \), as demonstrated by the data for \( e_n = 1 \) in Fig. 6. The open and filled circles represent data for frictionless disks at low and high \( \text{Re}_p \), respectively, and there is a large discrepancy (roughly a factor of 1.5) between the decay lengths for these two cases, as shown in the insets of Fig. 6(b) and (c). This is because for \( e_n = 1 \), all the energy is removed through fluid damping and the lower boundary. However, if we set \( e_n \leq 0.8 \) or use disks with Cundall-Strack friction with \( \mu = 0.6 \) (marked with asterisks), where energy is lost via slipping at grain-grain contacts, we find \( l_\phi \approx l_{v_x} \approx 1 \), independent of \( \text{Re}_p \). \( l_\phi \) is independent of increased friction, and \( l_{v_x} \) decreases slightly with increased friction (data not shown). The decay lengths for the particle-asperity model (stars in Fig. 6) behave similarly to frictionless disks when \( e_n = 1 \), since there is no explicit collisional dissipation in this case.

Figure 7 shows the velocity profiles of frictionless disks with \( e_n = 0.8 \) for varying \( \text{Re}_p \) with a value of \( \Theta \) that is above \( \Theta_e(\text{Re}_p) \) by roughly 10% (i.e., the system is mobilized, but near the boundary \( \Theta_e \)). We again note that \( l_{v_x} \) is nearly independent of \( \text{Re}_p \), in contrast to the case of \( e_n = 1 \) discussed above. However, the flights of mobilized grains depend strongly on \( \text{Re}_p \). The velocity profiles shown in Fig. 7 are typical of all simulations that include grain-grain dissipation, and they demonstrate two important features of grain motion at varying \( \text{Re}_p \). First, the height of the trajectories of mobilized grains increases with increasing \( \text{Re}_p \), then plateaus for \( \text{Re}_p > 20 \). This effect is caused by reduced fluid damping in the vertical direction (relative to grain inertia), so the height of grain trajectories is set by gravity and not fluid damping. Grains entrained in the flow will experience stronger fluid flow and thus acquire greater momentum that is then delivered to the bed upon collision. Second, as \( \text{Re}_p \) continues to increase, the maximum velocity achieved by any grain begins to decrease relative to the imposed fluid velocity. The dashed vertical line in Fig. 7 shows the free-stream fluid velocity, which is roughly 7.4 times \( v_0 \) (see Section II). The increasing slip velocity between the fluid and grains at moderate to high \( \text{Re}_p \) means that the grains never equilibrate to the fluid flow at high \( \text{Re}_p \) and instead feel a constant acceleration.

\[ v_x \propto \exp(-y/l_{v_x} D) \]

\[ y/l_{v_x} \text{ is the decay length for the particle-asperity model (stars in Fig. 6) behave similarly to frictionless disks when } e_n = 1, \text{ since there is no explicit collisional dissipation in this case.} \]

\[ l_{v_x} \text{ is independent of increased friction, and } l_{v_x} \text{ decreases slightly with increased friction (data not shown). The decay lengths for the particle-asperity model (stars in Fig. 6) behave similarly to frictionless disks when } e_n = 1, \text{ since there is no explicit collisional dissipation in this case.} \]

\[ v_x \propto \exp(-y/l_{v_x} D) \]

FIG. 6. The mobilized layer decays exponentially into the static bed with a decay length \( l_\phi \) for packing fraction and \( l_{v_x} \) for horizontal velocity that varies with grain properties. The packing fraction profile \( \phi \) is plotted on (a) linear and (b) semilog axes. Panel (c) shows the horizontal velocity \( v_x \) on semilog axes. The packing fraction and horizontal grain velocity profiles decay as \( \phi_0 - \phi \propto \exp(-y/l_\phi D) \) and \( v_x \propto \exp(-y/l_{v_x} D) \). Insets to (b) and (c) give \( l_\phi \) and \( l_{v_x} \) as a function vs \( e_n \). Open circles, stars, and asterisks represent frictionless disks, particle clusters (5-mers) with \( \mu_{\text{eff}} = 0.6 \), and disks with Cundall-Strack friction with \( \mu = 0.6 \) at relatively low \( \text{Re}_p \approx 7 \), respectively. Filled circles represent frictionless disks at higher \( \text{Re}_p \approx 50 \).

C. Tangential forces

We have demonstrated that the curves marking the static-to-mobile transition, and vice versa, are qualitatively insensitive to dissipation from intergrain collisions, except for the fact that \( \Theta_e^c \rightarrow \Theta_0^c \text{ as } e_n \rightarrow 0 \). Next, we ask how these boundaries are affected by the presence of tangential forces that arise from Cundall-Strack friction
FIG. 7. The horizontal velocity profiles of grains on (a) linear and (b) semilog axes for frictionless disks with $c_n = 0.8$. Increasing $R_{ep}$ causes an increase in the height of mobilized grains and a decrease in the grain velocity relative to the fluid velocity (i.e., grains do not equilibrate to the fluid flow). The inset of panel (b) shows that the decay length $l_v$ of grain flow into the bed is relatively independent of $R_{ep}$, which suggests that the grain motion in this region is dominated by the granular media and not the fluid.

and irregular grain shape. Figure 8 shows the results for systems using the particle-asperity model and the Cundall-Strack model. We find that the boundary between systems with and without grain motion is weakly dependent on the presence of tangential forces, with variation that is less than the typical scatter in Fig. 1(a).

Figure 9 shows the values of the plateaus $\Theta^l_c$, $\Theta^h_c$, and $\Theta^0_c$ as a function of the friction coefficient $\mu$ for disks with Cundall-Strack friction and the effective friction coefficient $\mu_{eff}$ for particle clusters (see Table I). Circles represent disks with Cundall-Strack friction, with a friction coefficient $\mu$. Other symbols correspond to $n$-mers with $n = 2$ (diamonds), 3 (triangles), 4 (squares), and 5 (stars). Figure 9 shows $\Theta^l_c$, which describes the yield strength of the strongest packing of grains in the limit where grain inertia is irrelevant. That is, since grain inertial effects are irrelevant, the grains can explore configuration space until they find a sufficiently stable state to resist the applied fluid force. Thus, one would expect that increasing friction would cause $\Theta^l_c$ to increase. Figure 9 also shows $\Theta^h_c$ versus $\mu$ or $\mu_{eff}$. As with $\Theta^l_c$, friction tends to increase the value of $\Theta^h_c$. For frictional disks (circles), $\Theta^h_c \approx 0.11$ for low friction and $\Theta^h_c \approx 0.18$ for high friction. Finally, Fig. 9 shows $\Theta^0_c$ versus $\mu$ or $\mu_{eff}$, which we measure by finding the minimum value at which a static system in one of the high $R_{ep}$ ensembles (i.e., $\Gamma = 0.02, 0.01$, or 0.005) fails. We find that $\Theta^0_c$ increases less than $\Theta_c$, from $\Theta^0_c \approx 0.27$ for low friction to $\Theta^0_c \approx 0.34$ for high friction. The plateau values $\Theta^l_c$, $\Theta^h_c$, and $\Theta^0_c$ for the particle-asperity model approach those for Cundall-Strack friction as $n$ increases. However, 2-mers and 3-mers tend to behave more like frictionless disks. Additionally, we emphasize that the change in the plateau values is roughly 1.5, which is comparable to the scatter in Fig. 1(a).

How do tangential forces affect the static-to-mobile
transition? A central insight of our previous work was that the static-to-mobile transition is governed by weakest-link statistics. That is, a static packing of grains will become permanently mobile at \( \Theta > \Theta_f \) for low \( \text{Re}_p \) and \( \Theta > \Theta_0 \) for high \( \text{Re}_p \) (in the equations below, we use \( \Theta_c \), which should be replaced by \( \Theta_0 \) in the large \( \text{Re}_p \) case). The statistics of when a particular system will fail are captured by considering the bed to be a composite system of \( M \) uncorrelated subsystems that fails if any of the subsystems fail. In this case, the cumulative distribution \( C_M(\Theta) \) for failure is related to that of a single subsystem \( C(\Theta) \) by

\[
1 - C_M(\Theta) = [1 - C(\Theta)]^M.
\]

By using a Weibull distribution for \( C(\Theta) \) [47–49]

\[
C(\Theta) = 1 - \exp \left[ \left( \frac{\Theta - \Theta_c}{\beta} \right)^\alpha \right],
\]

then \( C_M(\Theta) \) in Eq. (5) has the same form with \( \alpha_M = \alpha \) and \( \beta_M = \beta M^{-1/\alpha} \). As in our previous study [25], we find that this scaling holds. This means that Eq. (5) applies, and global failure is caused by the failure of a single member of a collection of uncorrelated subsystems. Thus, larger systems are more likely to fail near \( \Theta_c \) or \( \Theta_0 \) for small and large \( \text{Re}_p \), respectively, but systems that fail near these minimum values are very slow to become fully mobilized. Figure 10(a) shows the Weibull distributions of the excess stress \( \text{bar} \) \( \Theta_f - \Theta_0 \) required for a given system to fail. These distributions collapse when rescaled by their mean, as shown in the inset, with a shape parameter \( \alpha \approx 2.4 \). (This value is comparable to that measured in the frictionless, elastic case, where we found \( \alpha \approx 2.6 \) [25].)

The primary difference between the frictional and frictionless cases is in the effective system size \( M_{\text{eff}} = W_{\text{eff}} H_{\text{eff}} \). In systems with tangential forces, where \( W \) is larger than a few particles, we find that \( M_{\text{eff}} = W \), and

![Image](image-url)
the system height is nearly irrelevant. For frictionless disks, $H_{\text{eff}}$ is calculated by integrating the probability of failure over the depth of the system, which is equal to the fluid force profile. Thus, friction strongly suppresses the initiation of surface grain motion from deep beneath the surface. The open star symbols in Fig. 10 show the statistics of failure for ensembles of 5-mers with $\mu_{\text{eff}} = 0.6$ that were allowed to settle at $\Theta = 0.05$ at relatively high $Re_p \approx 750$ (with $\Gamma = 0.02$) for width $W/D = 5, 10, 20$, and 40, and depth $ND/W = 10, 20$, and 40. These 5-mers have $\mu_{\text{eff}} \approx 0.6$, which sufficiently suppresses failure deep in the bed such that the collapse is good for $M_{\text{eff}} = W$.

We also note that the failure of systems with tangential forces depends on preparation history in a way that the failure of frictionless systems does not. Specifically, systems with tangential contact forces that settle at a larger value of $\Theta$ tend to fail at larger values of $\Theta_f$. In frictionless simulations, we found no variation of the statistics of $\Theta_f$ with the value of $\Theta$ at which the system settled. The open stars shown in Fig. 10 settled at $\Theta = 0.05$, whereas the filled red stars settled at $\Theta = 0.1$. Settling at a larger value of $\Theta$ makes these systems slightly stronger on average, although they still have roughly the same minimum failure point.

IV. DISCUSSION

As we discussed in Section I, the goal of this paper is to understand how the boundaries marking the static-to-mobile transition, and vice versa, shown in Fig. 1(b), calculated for systems of frictionless elastic disks, change as grain-grain interactions are made to mimic natural systems, as well as how these boundaries relate to the collection of experimental and field data shown in Fig. 1(a). The primary insight from the results presented here is a potential explanation as to why the minimum $\Theta$ values for grain motion for a wide array different systems, with varying grain properties and fluid flow profiles, cluster around the same curve in Fig. 1(a). For low $Re_p$, the data in Table II and Fig. 4 show that $\Theta^b_c$ is independent of the fluid profile when the fluid stress at the top of a static bed is more than five times greater than a typical fluid force felt in the bulk, a condition which is certainly met in natural systems. The stability of these systems is dominated by the strength of the top layer, an effect which is reinforced by the presence of tangential forces, as shown in Fig. 10. $\Theta^b_c$ is independent of $\varepsilon_n$, and varies with friction by less than a factor 1.5, comparable to the scatter in Fig. 1(a). At high $Re_p$, $\Theta^b_h$ is independent of the fluid profile, as shown in Table II: independent of $\varepsilon_n$, as shown in Fig. 5(b); and only weakly dependent on friction, as shown in Fig. 9. In contrast, Table II shows that $\Theta^h_c$ depends strongly on the ratio of a typical fluid force above the bed to one at the surface. However, adding dissipation from grain-grain interactions significantly weakens this dependence, causing $\Theta^h_c \rightarrow \Theta^h_0$ as $\varepsilon_n \rightarrow 0$. In this study, grain-grain dissipation primarily represents losses due to intergran interactions mediated by a viscous fluid. Previous work has shown that $\varepsilon_n$ can be quite small, even approaching $\varepsilon_n = 0$, for stiff materials that would have $\varepsilon_n \approx 1$ in the dry case [41, 42]. Thus, our results suggest that in the presence of such dissipation, the stability of the bed at high and low $Re_p$ is largely independent of the details of the flow or the properties of the grains. Instead, the stability of the bed depends almost exclusively on the force felt by the top layer of the bed (i.e., the Shields number $\Theta$) and the subsequent grain dynamics, given a particular $Re_p$. (We again note that we have chosen to follow previous work in using $Re_p$ instead of the Stokes number $St \propto \frac{6}{7} Re_p$. Since the mass densities $\rho_g$ and $\rho_f$ are not varied in the case of sediment transport, $St$ and $Re_p$ are equivalent.) The behavior at low $Re_p$ is controlled by the yield strength of the strongest configuration of the top layer of grains, and the behavior at high $Re_p$ is controlled by the dynamics of grains bouncing, rolling, and sliding along a granular surface at constant acceleration.

To illustrate this picture, we consider a simple model, where a single grain becomes dislodged from a pocket in the bed, as depicted in Fig. 11. As we demonstrate, this simplified picture predicts particle dynamics that are independent of $Re_p$ in the limits of low and high $Re_p$, and it qualitatively captures how the plateau values of $\Theta$ depend on the coefficient of restitution and friction, as we have shown in Figs. 5 and 9. Assume the dynamics of an individual grain are governed by

$$ma_x = B(v_0 - v_x)$$
$$ma_y = -mg' - Bv_y.$$  \hspace{1cm} (7)

The time at which the grain becomes dislodged is purely a function of $\Theta$. For the grain in Fig. 11(a), stability will be lost when $\Theta = \cot \phi$. It can then land in one of the other pockets on the bed surface, labelled 1-3, depending on geometry, grain properties, and $Re_p$. In the limit of low $Re_p$, grain inertia is irrelevant, and the grain immediately equilibrates to the horizontal fluid velocity and falls at the settling velocity. Upon contact with the bed, the grain transfers negligible momentum. Thus, only geometry is relevant in the limit of low $Re_p$ (i.e., the dynamics are independent of $Re_p$), and $\Theta^b_c$ is set by yield strength of the most stable configuration of grains. $\Theta_c$ should be independent of the restitution coefficient in this limit, in agreement with Fig. 5(b), since the strong fluid damping yields collisions that are effectively inelastic. Friction or irregular particle shape should strengthen the system, thereby increasing $\Theta^b_c$ in agreement with Fig. 9.

In the limit of large $Re_p$, the grain will again be dislodged when $\Theta = \cot \phi$. In this limit, the fluid is moving sufficiently fast that the grain will never equilibrate to the fluid and will feel a constant acceleration $a_x = \Theta g'$ (i.e., the dynamics are again independent of $Re_p$). For the grain to land in one of the pockets, the grain must be stable in that pocket at the given $\Theta$, but the dynamics of whether it is able to stop in that pocket depend on the...
assume the grain travels a horizontal distance \( d \) the potential well (i.e., the sharp corners shown here) and lodged and lands in a stable well. We neglect the shape of configuration where a grain governed by Eq. (7) become dis-trapped, the critical Shields numbers for this process are assuming a grain must fall half its diameter to become

\[ \Theta_c \approx \frac{1}{2} \text{ for disks}. \]

amount of momentum acquired during the collision with the bed. If the grain travels too far, it could bounce out of the pocket and keep traveling, or it could reshape the pocket by dislodging other grains in its collision with the bed. This process also depends on friction, since grains that roll without slipping will acquire less momentum in a given distance. Specifically, applying simple kinematic equations for constant acceleration to the situation in Fig. 11(b) and assuming

\[ d_1 = d_2 = D, \]

frictionless grains will have velocity \( \sqrt{2BDv_0/m} \) and very frictional grains will have velocity \( \sqrt{2BDv_0/[(\beta_f + 1)m]} \) after traveling distance \( d_1 = D \), where \( \beta_f \) is the geometrical prefactor in the moment of inertia (i.e., 1/2 for disks). Assuming free fall at constant acceleration \( a_g = -g \) over \( d_2 = D \), and assuming a grain must fall half its diameter to become trapped, the critical Shields numbers for this process are \( \Theta_c \approx 0.34 \) for the frictionless case and \( \Theta_c \approx 0.45 \) for the high friction case. The high friction limit is roughly 1.3 times bigger than that for the frictionless case, which roughly agrees with Fig. 9.

An additional effect shown in Fig. 7 at high \( \text{Re}_p \) involvs the height of the trajectories of mobilized grains, which can travel above the bed and experience stronger fluid flow. This effect is negligible in the limit of low \( \text{Re}_p \). However, in the case of a mobilized bed in the high \( \text{Re}_p \) limit, mobilized grains feel a stronger constant acceleration, will acquire greater momentum, and will deliver more energy to the bed upon collision. The increasing height of grain trajectories and momentum of collisions are the primary causes for the hysteresis at large \( \text{Re}_p \), i.e., \( \Theta_h > \Theta_i \). As \( e_n \) is reduced, grains will not rebound significantly above the bed, and the dynamics are again dominated by a grain rolling or sliding along the surface. This will reduce the hysteresis, in agreement with Fig. 5(b). We note that for Aeolian transport of desert sand \( [32, 34] \), where the driving fluid is air instead of water, intergrain collisions will likely have effective restitution coefficients that are much larger. Our results suggest that hysteresis should persist in the case of Aeolian transport.

Here, we have focused on two-dimensional (2D) simulations, which, in principle, can differ from three-dimensional (3D) flows. The potential energy landscape for a grain on top of a 3D granular bed will differ from that for the 2D case. For example, spherical grains can roll or slide between other grains in a way that disks cannot. Additionally, the linear, Stokes-like drag is not accurate at moderate to high \( \text{Re}_p \), since inertial effects will begin to dominate yielding drag that is quadratic in the fluid-grain velocity difference. We expect this effect to be most important in the transition regime, roughly \( \text{Re}_p \sim 1 - 100 \), where inertial drag first becomes important and grains are more likely to equilibrate to the fluid flow. At high \( \text{Re}_p > 100 \), we expect the fluid force felt by grains near \( \Theta_c \) to be roughly constant acceleration given by \( \Theta g' \). Entrained grains in the limit of high \( \text{Re}_p \) will never equilibrate to the fluid flow, as shown in Fig. 7, so the form of the drag law is considerably less important. The main results presented here concern the behavior in the limits of low and high \( \text{Re}_p \), and we expect that the simplified form of the drag law will not significantly alter the results in these regimes. In future work, we will consider a more realistic drag law, which will be important in the transitional regime, as well as 3D simulations.

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