Stress Fluctuations for Continuously Sheared Granular Materials

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Experiments on continuously sheared granular materials (glass spheres) with diameters $1.0 \le d \le 5$ mm show large fluctuations in the normal stress, $\sigma(t)$. Experiments are carried out in an annular Couette geometry for rotation rates $0.003 < \dot{\theta} < 1$ rad/s. Power spectra from $\sigma(t)$, $P(\omega)$, show rate invariance in the fluctuations: $\dot{\theta}P(\omega)$ is a function only of $\omega/\dot{\theta}$, independent of $\dot{\theta}$. $\dot{\theta}P(\omega)$ depends relatively weakly on d. The distributions of stresses ρ_σ are similar to recent predictions for static arrays, but the width of ρ_σ varies only weakly with d, suggesting stronger spatial correlation effects than expected from theory. [S0031-9007(96)01367-1]

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Granular materials exhibit a large and often poorly understood set of dynamical states, ranging from rapidly moving dilute flows to slowly deforming dense ones [1–4]. These materials are key to a number of important technologies [5]. An increased understanding of these systems could provide insight into other related phenomena, including such geophysical processes as earthquakes, avalanches, and dune formation, as well as such manmade phenomena as traffic flow or gridlock. Many recent studies of granular materials have focused on the dilute regime, for instance, in rapidly shaken systems [3]. However, the dense granular materials considered here, which are particularly common because gravity and dissipation causes granular materials to coalesce, are also poorly understood at a fundamental level.

A large literature exists, dating to the time of Coulomb [2,6–8], which deals with continuum models for dense granular materials. These models attempt to capture the mean local properties of the material. Modern versions typically assume elastoplastic behavior which means, roughly speaking, that under normal stress, σ, very little deformation occurs unless the shear stress is strong enough to cause large scale frictional slipping, leading to irreversible changes in grain configurations. These models imply rate independence: a uniform rescaling of the strain rates leaves the stresses unchanged. Continuum models in general tacitly assume that there are length scales above which fluctuations are negligible. However, to our knowledge, this assumption has not been justified, and it is this point which the present experiments address.

On the microscale of a few grain diameters, particles in a dense granular material form disordered frozen spatial configurations, which suggests that force fluctuations should be short range. However, the application of stress to a granular system leads to stress chains. These are approximately linearly aligned sets of grains which carry disproportionally large amounts of the total force [9], and can extend over many grain diameters (inset of Fig. 1, or in Liu *et al.* [10] or Behringer [11]). The statistical and dynamical properties of these chains are only beginning to be understood [12]. Several recent experiments [13,14] suggest that inhomogeneities and fluctuations play an important role in the time evolution of dense granular flows, and stress chains likely play an important part in this complexity. Moreover, in such flows, individual grains move relatively slowly, so that the rapid temporal averaging of stresses which occurs in Newtonian fluids does not take place.

Recently, Coppersmith *et al.* [10] have developed a new model (the "q model") and carried out experiments to characterize the statistical properties of static stress distributions. The q model can be envisaged in a simple 2D case as a regular packing of disks supported from below. Vertical forces are transmitted from a disk (disk A) to its two supporting neighbors below. The two supporting disks carry a random fraction (q and $1 - q$) of the other vertical forces (gravity, forces from disks above) on disk

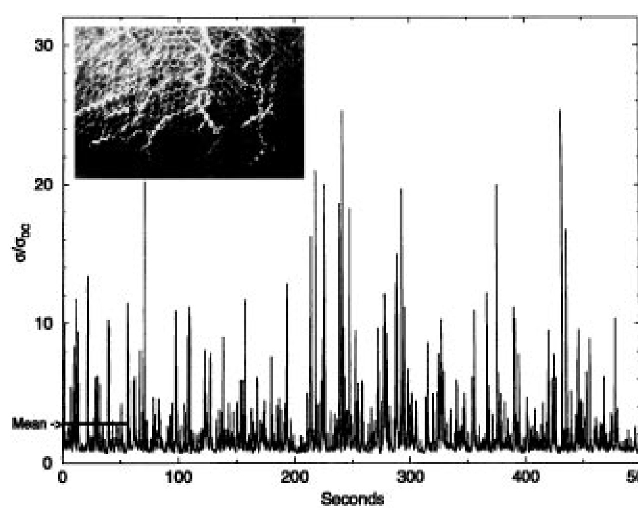

FIG. 1. Typical time series for the normal stress as a function of time, for $d = 4$ mm and $\dot{\theta}/(2\pi) = 20$ mHz. The data are normalized by the dc stress which we measure in the absence of shearing. The horizontal line indicates the mean. Inset: stress chains visualized by photoelasticity. Brighter disks are subject to larger stresses than darker disks.

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A. This model predicts that the distribution of forces on individual grains in a static array subject to gravity should vary as $\rho_f = f_0^{-1}(f/f_0)^2 \exp(-f/f_0)$ in 3D. This model is of considerable interest because it provides new insight into the propagation and fluctuation of stresses in granular materials. The present experiments involve dynamical vector forces rather than static ones, so the q model should be considered here as a useful guide.

The present work takes a quantitative step towards understanding spatiotemporal fluctuations for a dense slowly evolving granular system. By varying the grain size and the height of the layer, we can test for spatial averaging of the fluctuations. By measuring time series at relatively high sampling rates, we can look for the dynamics of the fluctuations.

Our apparatus is shown in the inset of Fig. 2; an annular gap contains the granular material (approximately monodisperse glass spheres with diameters $1.0 \leq d \leq 5$ mm). A rotating upper ring of width $w = 2.5$ cm and mean diameter $D = 35.6$ cm provides continuous shearing. The mass, $M = 8.1$ kg, of the ring and attached structure is supported by the granular material. M is always greater than the mass of the granular material, which is between 0.44 and 1.8 kg. The shearing ring rotates on a shaft and precision bearing which allows free vertical displacement. We vary the height of the layer between $1.0 \leq h \leq 4.1$ cm. To guarantee that the particles are actually sheared, we glue a layer of particles to the shearing ring. The smooth curved sidewalls and smooth flat base of the annulus remain at rest. We measure the absolute normal stress at the bottom of the layer with a capacitive pressure stress transducer incorporated into an ac bridge, similar to that used by Baxter *et al.* [14]. The calibrated transducer is flush with the inside of the base, and centered radially in the annulus. It has an active area of circular cross section (area $= 0.8$ cm^2), and a frequency response which is flat up to about 2 kHz. A dc motor and drive assembly provide torque to rotate the shearing ring. We also measure θ, the angular displacement of the ring, by means of a three-turn resistive potentiometer that has a rubber wheel on its axis making frictional contact with the shearing ring. We measure as a function of time (a) the capacitance of the stress transducer, (b) the resistance of the pot, and (c) the current from the fixed-voltage drive motor. Both the power provided by the motor and $\dot{\theta}$ show only small temporal variations compared to fluctuations in the normal stress at the bottom of the layer. Hence, to a reasonable approximation, the measurements are carried out at constant torque. In all runs, we first determine the ac bridge setting corresponding to an empty cell without the shear ring or spheres in place. We fill the cell by pouring in an approximately monodisperse sample of glass spheres, which we level. We then place the shearing ring on top of the sample and measure the resulting stress, σ_{dc}, which provides a typical value about which major stress variations may occur during the shearing process. Although we cannot see more than the top row of spheres during an experiment, the typical behavior for this type of flow consists of a shear layer near the ring such that grains near the top are in motion and slide over lower grains which remain at rest.

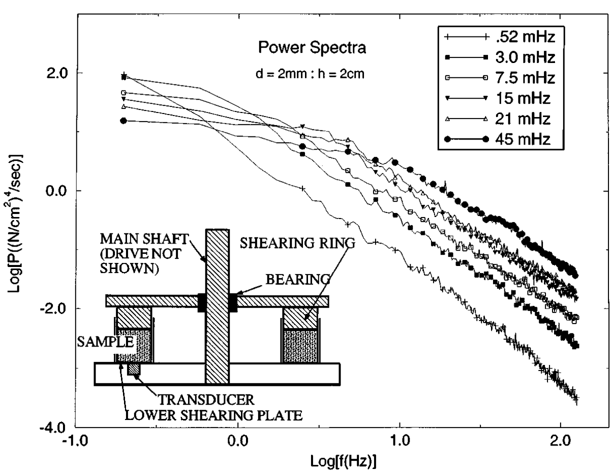

FIG. 2. Representative spectra for $d = 2$ mm. The rotations rates, $\dot{\theta}$ (normalized by 2π) of the shearing ring are noted in the upper corner. At high ω, the spectra vary as ω^{-2}, and more weakly with ω at low ω. Inset: schematic of vertical cross section of the apparatus.

One of the most notable features of these data is that extremely large scale fluctuations occur, as in Fig. 1, which shows data for $\sigma(t)/\sigma_{\mathrm{dc}}$. Note that $\sigma(t)$ can have fluctuation events which are an order of magnitude greater than σ_{dc}. The implications are significant: specifically, localized time-dependent stresses can be very large—far larger than the mean, which is indicated by the horizontal line.

Figure 2 shows typical power spectra, $P(\omega)$, obtained from $\sigma(t)$ (in dimensioned stress units). For a given $\dot{\theta}$, the spectra vary as $1/\omega^2$ at large ω. At low ω, the spectra become flatter, with a typical variation $P(\omega) \propto \omega^{-\alpha}$ with $\alpha \simeq 0.6$. The time series and spectra are somewhat similar to molecular dynamics calculations by Savage [15]. An interesting question is the dependence of the spectra on the shearing rate, $\dot{\theta}$. To the extent that the shearing occurs slowly compared to the collective relaxation rate of the system, we would expect, at least on average, rate invariance if time is scaled by $\dot{\theta}$. (An analogy is an old-fashioned record whose content, if not its aesthetic properties, do not depend on the playing speed for not-too-fast rates.) For a rate-independent process, $\dot{\theta} P(\omega)$ should depend only on $\omega/\dot{\theta}$. Figure 3, which shows two examples of data for 1.5 and 4 mm particles in scaled form, indicates very clear rate scaling of the spectra over several orders of magnitude in $\dot{\theta}$. This rate independence is an interesting and novel feature, since it applies to the *fluctuating* component of the stress rather than to the *mean* properties which are modeled by continuum theories.

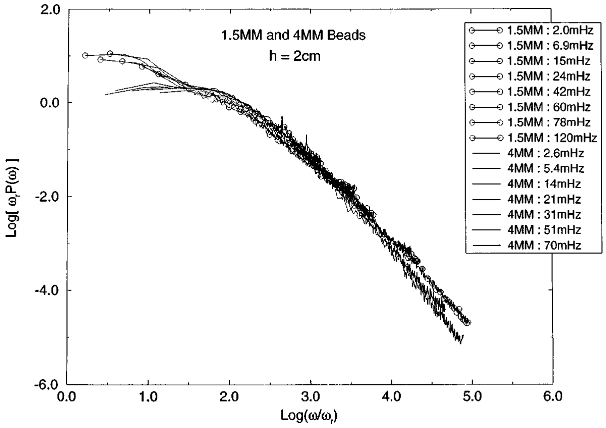

FIG. 3. Scaled power, $\dot{\theta}P(\omega/\dot{\theta})$, vs scaled frequency, $\omega/\dot{\theta}$, for $d = 1.5$ and 4 mm, demonstrating rate independence. The scaled spectra for different grain sizes are similar over the parameter ranges considered here, except perhaps at low ω.

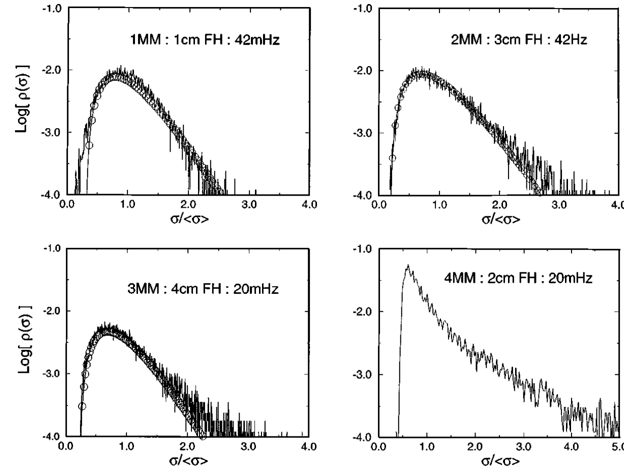

FIG. 4. Distributions for σ, normalized by the mean stress $\langle\sigma\rangle$, for different grain sizes but comparable rotation rates. A small positive quantity has been added to the ρ's in order to keep the logarithm finite. The circles indicate least squares fits to the form $\rho = A(\sigma - \sigma_{\text{offset}})^2 \exp(-\sigma/\sigma_0)$, as suggested by the analysis of Coppersmith *et al.* [10]. The data for $d = 4$ and 5 mm did not fit well to the model because of the relatively sharp rise for small σ.

An inspection of these scaled spectra, Fig. 3, shows that they do not depend strongly on particle size. An obvious question is what is the physical process associated with the change in exponent of $\dot{\theta}P(\omega/\dot{\theta})$? One natural frequency scale is associated with the time for the flowing layer to move by a grain diameter, $\omega_d \sim R\dot{\theta}/d$, where R is the radius of the shear ring. For $d = 2$ mm, $\omega_d/\dot{\theta} \simeq 100$, which is the approximate location of the crossover frequency ω_c. This suggests that ω_c might vary as d^{-1}. Although there is a trend for ω_c to decrease with increasing d, the variation does not appear, within the scatter, to be as strong as $\omega_c \propto d^{-1}$, and there is also clear h dependence in ω_c. Hence, the details which control this feature will require additional investigation.

The time-varying force at the transducer can be viewed as a measure of forces for an ensemble of different arrangements of the grains (e.g., Edwards and Oakshot and Mehta [16]) which are continuously being stirred by the shearing process. As long as the observation time is sufficiently long, the ensemble should be well sampled, and a statistical approach justified. Hence, ρ_σ, a distribution of the stresses from the present experiments, without regard to time, might resemble the predictions of the q model [10]. Figure 4 shows that this is at least approximately the case. In this figure, we present representative examples of ρ_σ vs σ. For most cases, ρ is reasonably well described by $\rho(\sigma) = A(\sigma - \sigma_{\text{offset}})^2 \exp[-(\sigma - \sigma_{\text{offset}})/\sigma_0]$, where σ_{offset} may account for small shifts in the zero of stress. Interestingly, the distributions for the largest particles, $d = 4$ and 5 mm are qualitatively least like the q model, although they do fall off at large σ more or less exponentially. (Distributions for $d = 5$ mm are qualitatively similar to those for $d = 4$ mm, and are not shown.)

An important point concerns the variations of stress distributions with d and h. In particular, as we change the particle diameter from 1.0 to 5 mm, the ratio of the transducer area to the particle cross section changes from 100 to 4. This is roughly representative of the number of grains in contact with the transducer surface at one time. We might expect that the averaging effect of many grains for the smaller particles would lead to smaller fluctuations. The effectiveness of the averaging on reducing the fluctuations will depend on the horizontal spatial correlations of the forces due to individual particles which act collectively on the transducer. For instance, if N grains in the plane of the transducer each have an independent distribution $\rho_f = f_0^{-1}(f/f_0)^2 \exp(-f/f_0)$ (in the q model, such grains are statistically uncorrelated) then the distribution for their collective force F is $\rho_F = [f_0(3N-1)!]^{-1}(F/f_0)^{3N-1}\exp(-F/f_0)$. Matching to the effect of changing the particle size at fixed detector area in the experiment requires that the mean force $\bar{F}$ from the particles be constant. (This last condition is more complex in practice, because some of the normal load is supported by friction between the grains and vertical walls.) Then, $f_0 = \bar{F}/3N$, and the variance for ρ_F is $\bar{F}(3N)^{-1/2}$. This would imply roughly a fivefold variation in the width of the distributions for particle sizes ranging from 1.0 to 5 mm. By contrast, the width of the experimental distributions do not show such a strong variation with particle size, even though the qualitative shapes of the distributions change with d. This suggests that correlation effects not present in the q model are important. More specifically, data for the variance $\Sigma^2 \equiv \langle(\sigma - \bar{\sigma})^2\rangle$ (to be presented elsewhere) do not show any obvious systematic dependence on particle size, on height of the granular layer h, or on $\dot{\theta}$. [Constancy with respect to $\dot{\theta}$ is, in fact, another indication of rate independence, since $\Sigma^2 = \int P(\omega)\,d\omega = \int \dot{\theta}P(\omega)\,d(\omega/\dot{\theta})$.]

Interestingly, the distributions are most like the q model when the number of particles in contact with the transducer is relatively large. It is important to note, however, that the q model was not constructed to describe the dynamically evolving vector forces present in this experiment.

The apparent lack of dependence of Σ^2 on d and h suggests a picture in which a significant fraction of the force is carried by chains of length $\sim h$. At least for large F, the distribution of forces delivered to the transducer would then be determined by a small number of stress chains between the transducer and the shearing ring, and not the total number of grains contacting the transducer.

To conclude, we have shown that force fluctuations in temporally evolving dense granular materials can be quite large on local spatial scales and can occur over a broad range of time scales. Notably, the spectra for the stress show rate independence. More broadly, the scaled spectra are similar for all particle sizes and fill heights considered here, except perhaps at the lowest scaled frequencies. The distribution of stresses is similar to recent predictions for static force distributions. However, the width of the force distributions shows little systematic variation with particle size, at least on the scales which we have achieved in these experiments. This suggests that fluctuational effects may be of real significance in the design of practical granular flow devices.

Several caveats apply, however. Although we find that the variance is at best weakly dependent on d, and h, we can vary both d and h by only a factor of 4 to 5. Hence, the long range scaling behavior of the fluctuations remains an open question. However, in this regard, stress chains in static experiments can easily be as long as 10^2 grains [11]. The role of the boundaries in this type of system is not well understood, and additional investigation is in order. Because detectors of the type used here are not uniformly sensitive to force over their surfaces (i.e., the response falls off near the detector edge) some details of the stress distributions may change with the detector. This is unlikely to perturb the rate independence we observe, however.

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